Neutron stars:

stationary, quasi-stationary and oscillation models

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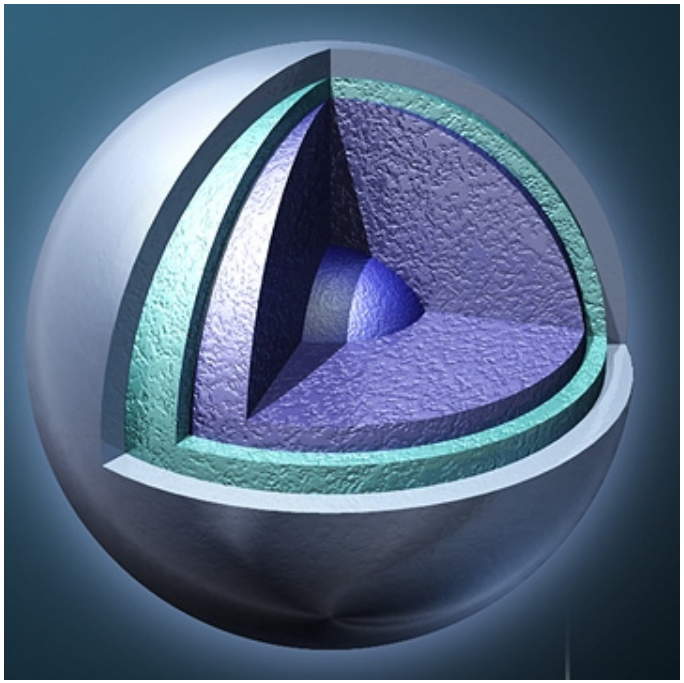
Plan of the lecture

1. Isolated rotating stars
2. Magnetic field
3. Quasi-stationary models of compact binaries
4. Evolution of a single neutron star: oscillations

The complex physics of neutron stars

The description of neutron stars involves many different fields of physics, with overall conditions that can hardly be tested on Earth:



- cold, highly asymmetric nuclear matter,
- very strong gravitational field (last stage before black hole),
- intense magnetic field, up to $\sim 10^{17}$ G,
- rapid rotation, implying relativistic fluid velocities.

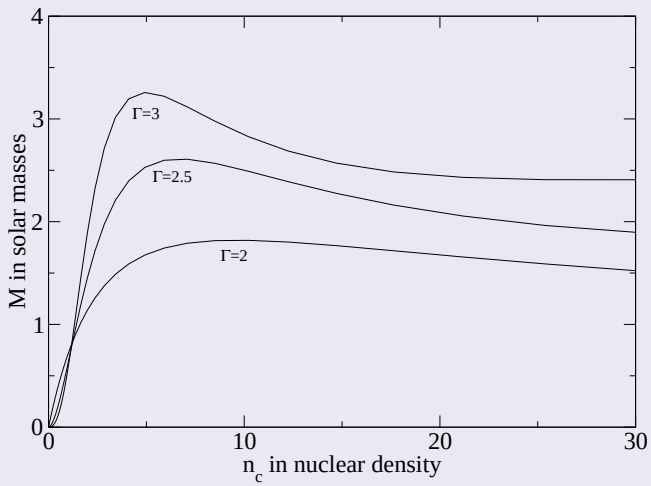
$\Rightarrow$need for theoretical models, often involving numerical simulations

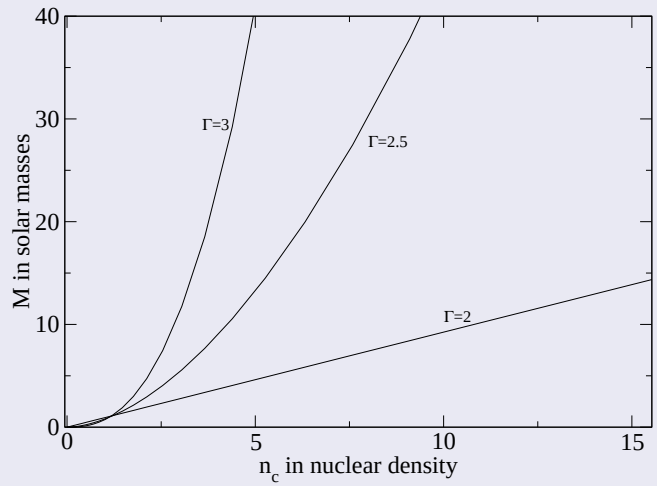
Need for GR

Influence of general relativity (GR) can be measured by the compactness ratio:

$$\mathcal{C} = \frac{GM}{Rc^2}. \qquad \mathcal{C} = 0.5 \iff \text{black hole.}$$

GR makes qualitative difference:



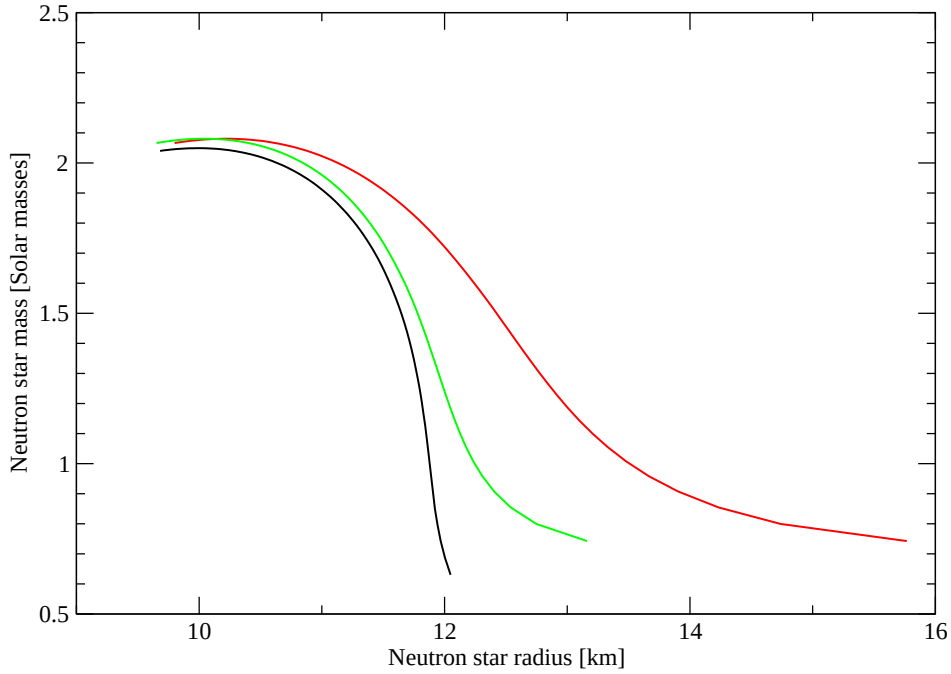


No maximal mass in Newtonian theory!

⇒General Relativity is absolutely necessary...

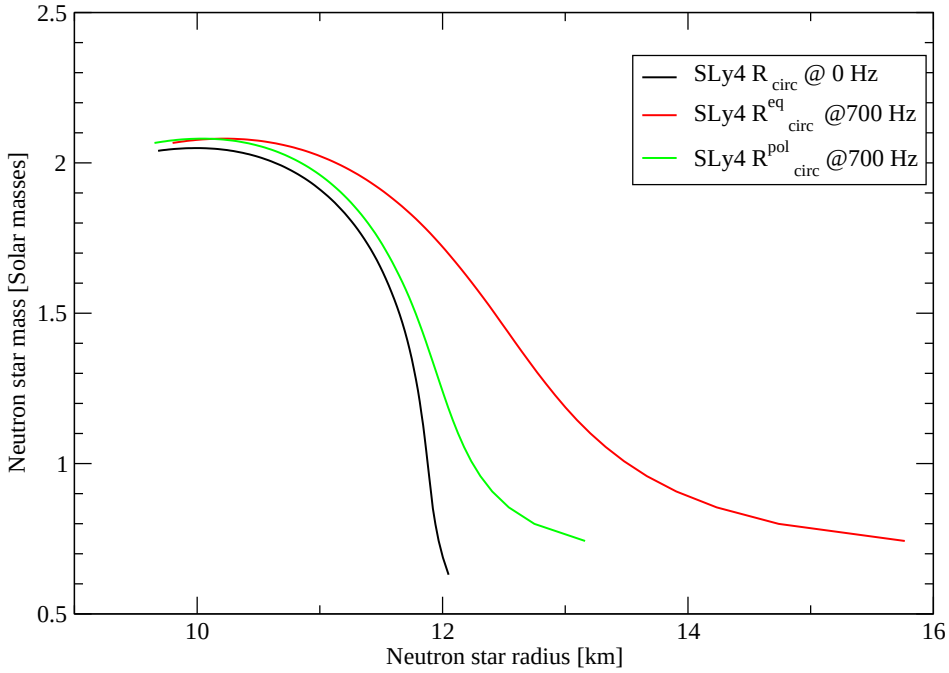
Need for rotation?

Three different EoSs ...



Need for rotation

One EoS : SLy4 Douchin & Haensel (2001)



Rotation is important when dealing with radii. $\Rightarrow$Need for precise definitions...

Brief history

rotating neutron star models

- Hartle & Thorne (1968) : slow rotation approximation,
- Bonazzola & Maschio (1971) : Lewis-Papapetrou coordinates,
- Wilson (1972) : differentially rotating stars
- Butterworh & Ipser (1975) : Bardeen-Wagoner formulation,
- Friedman *et al.* (1986) and Lattimer *et al.* (1990) : realistic EoSs,
- Bocquet *et al.* (1995) : (electro)magnetic field,
- ...

Some codes:

- Komatsu *et al.* (1989) $\Rightarrow$KEH,
- Bonazzola *et al.* (1993) $\Rightarrow$ `rotstar` (LORENE)
- Stergioulas & Friedmann (1995) $\Rightarrow$ `rns`
- compared in Nozawa *et al.* (1998)
- Ansorg *et al.* (2002) $\Rightarrow$ AKM

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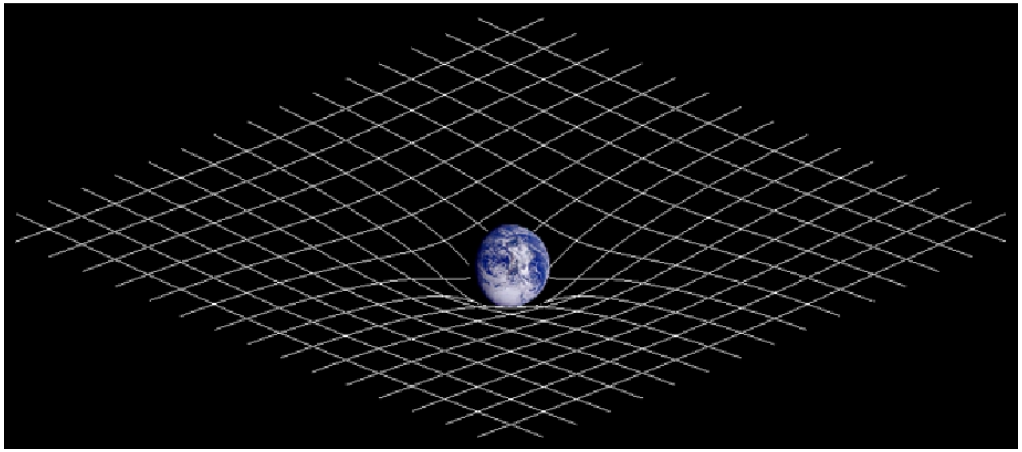
General Relativity in a nutshell



Spacetime is curved by the presence of energy/matter.

- The metric (4x4 symmetric matrix depending on the point coordinates): fundamental object to describe spacetime properties
- 2^{nd} derivatives = curvature
- Einstein equation relates metric (and derivatives) to energy/matter content: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
- Think of as Newton law for gravity + (gravitational) wave equations (hyperbolic)

Tolman-Oppenheimer-Volkov system

The metric $g_{\alpha\beta}$ in spherical symmetry and static depends only on r:

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -N^2c^2\mathrm{d}t^2 + A^2\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

The matter is assumed to be a perfect fluid and is given by its energy-momentum tensor $T^{\alpha\beta}$:

$$T^{\alpha\beta} = \left(e + \frac{p}{c^2}\right)u^\alpha u^\beta + pg^{\alpha\beta}.$$

Defining $m(r)$ and $\Phi(r)$ from $A = \left(1 - \dfrac{2Gm}{rc^2}\right)^{-1}$ and $N = \exp\left(\Phi/c^2\right)$, Einstein and hydrostatic equations write:

$$\begin{aligned}
\frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi r^2 e \\
\frac{\mathrm{d}\Phi}{\mathrm{d}r} &= \left(1 - \frac{2Gm}{rc^2}\right)^{-1}\left(\frac{Gm}{r^2} + 4\pi G\frac{p}{c^2}r\right) \\
\frac{\mathrm{d}p}{\mathrm{d}r} &= -\left(e + \frac{p}{c^2}\right)\frac{\mathrm{d}\Phi}{\mathrm{d}r}
\end{aligned}$$

Adding the equation of state (EoS) integrate from the center (parameter p_c) out to the surface, where it is matched to vacuum solution.

$p(e)$ (EoS) $\iff$ $M(R)$ (see Nicolas Chamel's talk)

Isolated rotating stars

Symmetries - Metric

To simplify the problem in GR: use of symmetries $\Rightarrow$Killing vector fields:

- stationarity
- axisymmetry

In adapted coordinates, the metric depend only on (r, θ) and can take the form (quasi-isotropic gauge):

$$ds^2 = -N^2 dt^2 + A^2 \left(dr^2 + r^2 d\theta^2\right) + B^2 r^2 \sin^2\theta \left(d\varphi - \omega\, dt\right)^2 ,$$

with the requirement of circularity condition for matter:

- no meridional (e.g. convective) currents,
- no toroidal magnetic field,

This is quite different from the Schwarzschild gauge used for the TOV system, in spherical symmetry.

Einstein equations

In quasi-isotropic gauge (+maximal slicing), Einstein equations turn into a system of four coupled non-linear elliptic PDEs:

- $\Delta N = \sigma_1$,
- $\Delta \omega = \sigma_2$,
- $\Delta (NB) = \sigma_3$,
- $\Delta (NA) = \sigma_4$.

Each σ_i contains terms involving matter and non-linear metric terms of non-compact support.
$\Rightarrow$Contrary to spherical symmetry no matching to any known vacuum solution is possible (no Birkhoff theorem).
$\Rightarrow$Only boundary condition at $r \to \infty$: flat metric.

Equilibrium & EoS

In the stationary, axisymmetric and circular case, the conservation of energy-momentum turns into a simple first integral of motion:

$$H + \log N - \log \Gamma = \text{const.}$$

$H = \log\left(\frac{e+p}{n_B m_0 c^2}\right)$: pseudo-enthalpy, Γ : Lorentz factor.
$\Rightarrow$ equilibrium condition.
The system is closed by the description of microscopic properties of the fluid: the equation of state:

$$p(n_B), e(n_B) \text{ or } p(H), e(H)$$

for cold nuclear matter at β-equilibrium.

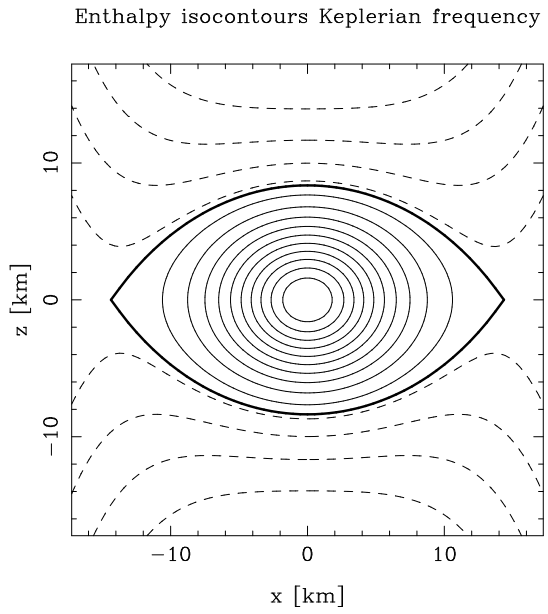
Global quantities

In general relativity, many quantities (*e.g.* mass, radius, ...) are gauge-dependent. Use of symmetries and physical definitions to get observationally relevant information:

- gravitational mass: Komar mass (stationarity) and ADM mass (asymptotic flatness) are equal in our case. They can be computed from the asymptotic behavior of N or as an integral over the support of $T^{\mu\nu}$.
  $\Rightarrow$ mass felt by a particle orbiting around the star.
- baryon mass: just counting the total number of particles... gauge independent.
- circumferential radius: ds integrated (with the metric) over a closed line (circumference), at the equator or passing through the poles.
- angular momentum (axisymmetry) : asymptotic behavior of ω

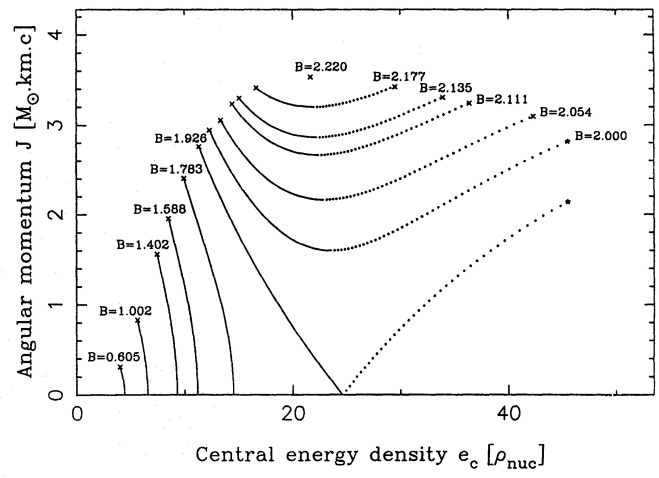
Examples

Salgado *et al.* 1994



- Rotation frequency limited by mass-shedding limit: matter leaving the star at the equator, due to centrifugal force.
- Impossible to describe in slow-rotation limit.

- Existence of supermassive sequences: for given baryon mass, there exist rotating solutions, but no static one.



Magnetic field

Motivations

Theoretical

- conservation of magnetic flux: 1 G for an O-type star of $\sim 10 R_\odot \Rightarrow \sim 10^{12}$ G.
- More if magneto-rotational instability in CC-SN $\Rightarrow$magnetars.
- influence on structure?

Observational

- Magnetic slowdown measured through the spin-down $\dot{P}$ gives values of $B_{\rm pole}$ up to 10^{16} G,
- magnetars could represent as much as 10% of all pulsars (Muno *et al.* 2008);
- they can produce very strong X- and γ-ray bursts, from the glitch-like rearrangement of the crust, in which magnetic field is pinned (*e.g.* Dec. 2004 with SGR 1806-20).

Model

Rotating stationary axisymmetric star + :

- independent currents j^α generating the electromagnetic field (Maxwell equations),
- stationary equilibrium with Lorentz force,
- symmetry conditions $\Rightarrow$ magnetic moment aligned with the rotation axis (no emission of electromagnetic waves),
- magnetic field is supposed to be purely poloidal.

Define:

- fields, as observed by the Eulerian observer

$$E_\mu = F_{\mu\nu}\, n^\nu,$$
$$B_\mu = -1/2\, \epsilon_{\mu\nu\rho\sigma}\, n^\nu\, F^{\rho\sigma}.$$

- global charge Q and magnetic dipole moment $\mathcal{M}$ deduced from asymptotic behavior of E_μ and B_μ, respectively.

Magnetic field equations

- $D_\nu F^{\mu\nu} = \mu_0 j^\mu$ can be written

$$\begin{aligned} \Delta_3 A_t &= S_1\left(j^\mu, g_{\mu\nu}, A_\mu, +\text{derivatives}\right) \\ \tilde{\Delta}_3\left(\frac{A_\varphi}{r\sin\theta}\right) &= S_2\left(j^\mu, g_{\mu\nu}, A_\mu, +\text{derivatives}\right) \end{aligned}$$

- Lorentz force appears in the first integral of motion

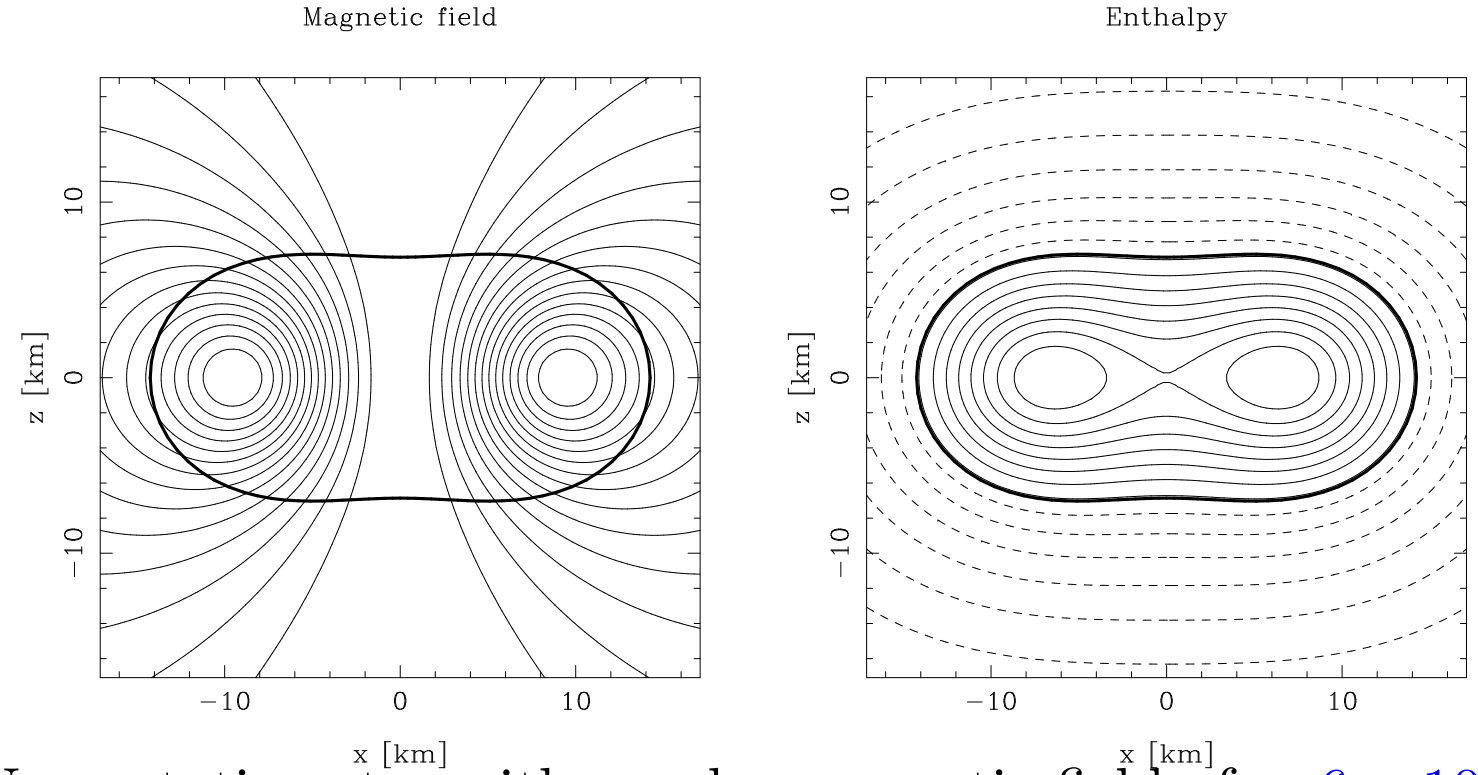
$$H(r,\theta) + \nu(r,\theta) - \ln\Gamma(r,\theta) - \int_0^{A_\varphi(r,\theta)} f(x)\mathrm{d}x = 0$$

  f arbitrary function such that $j^\varphi - \Omega j^t = (\varepsilon + p) f(A_\varphi)$,
- perfect conductor relation inside the star $A_t = -\Omega A_\varphi + C$.
- Non-isotropic energy-momentum tensor

$$T^{\mathrm{EM}}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}g_{\mu\nu}\right).$$

Deformation and magnetization

Chatterjee *et al.* (2015)



Non-rotating star with a polar magnetic field of $\sim 6 \times 10^{17}$ G.

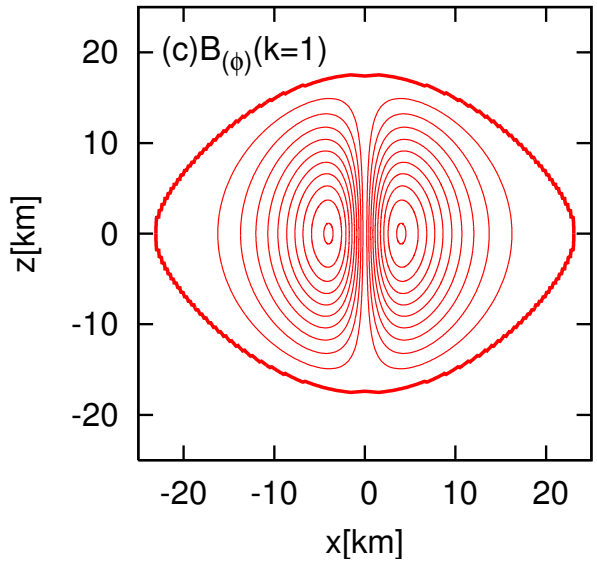
$\Rightarrow$Stars with magnetic field can support more mass

$\Rightarrow$Taking into account magnetic field in the EoS and magnetization in the equations : small effect (to be confirmed with other EoSs).

Poloidal / toroidal $\vec{B}$ field

$\Rightarrow$ Models with poloidal magnetic field are unstable (*e.g.* Ciolfi *et al.* 2011)

- Models with purely toroidal field have been built (Kiuchi & Yoshida 2008, Frieben & Rezzolla 2012)
- they show prolate shapes
- unstable, too ...



from Kiuchi & Yoshida (2008)

$\Rightarrow$ mixed configurations, with stratification... (Yoshida *et al.* 2012).

Quasi-stationary models of compact binaries

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Quasi-stationarity: helical Killing vector field

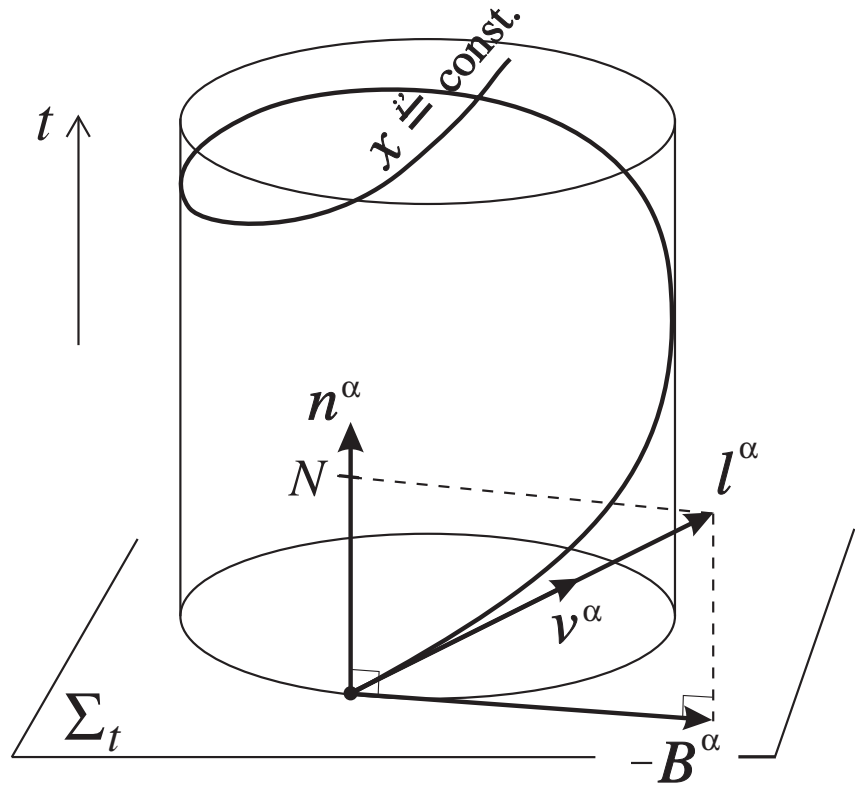
Binary systems of compact objects emit gravitational waves $\Rightarrow$ no equilibrium

BUT...

Discarding gravitational wave emission allows for *quasi*-stationary solutions.

Symmetry generated by a helical Killing vector field:

$$\partial_t - \Omega \partial_\varphi$$



This assumption is valid when both object are far apart.

Conformal-Flatness Condition

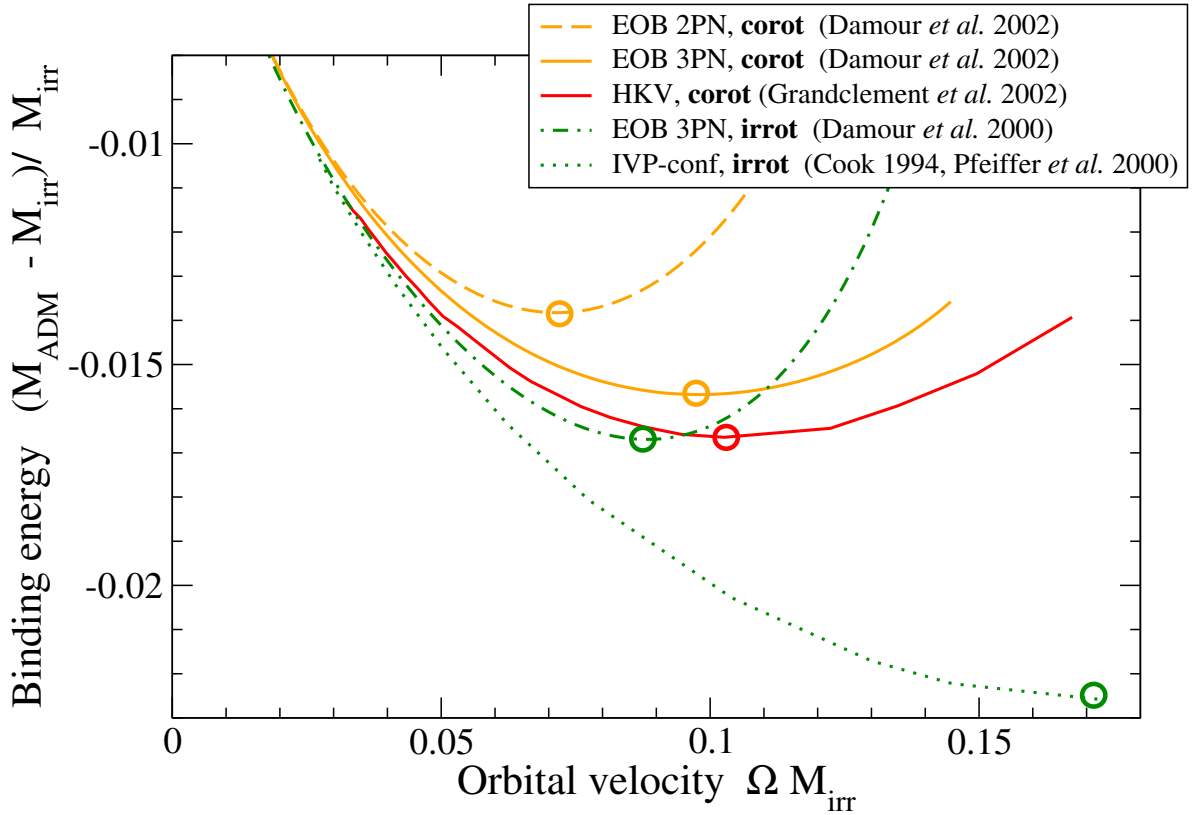
4 A sufficient condition to have no gravitational waves is to set the spatial part of the metric γ_{ij} to be conformally flat:

$$g_{\mu\nu} = \left(\begin{array}{c|c} g_{tt} & g_{tj} \\ \hline g_{it} & \gamma_{ij} \end{array}\right), \text{ with } \gamma_{ij} = \psi^4 f_{ij}.$$

Properties:

- approximation to General Relativity,
- exact in spherical symmetry (*e.g.* Schwarzschild black hole)
- no gravitational waves
- no Kerr black hole, but a rotating black hole not solution of Einstein equation.

Innermost Stable Circular Orbit

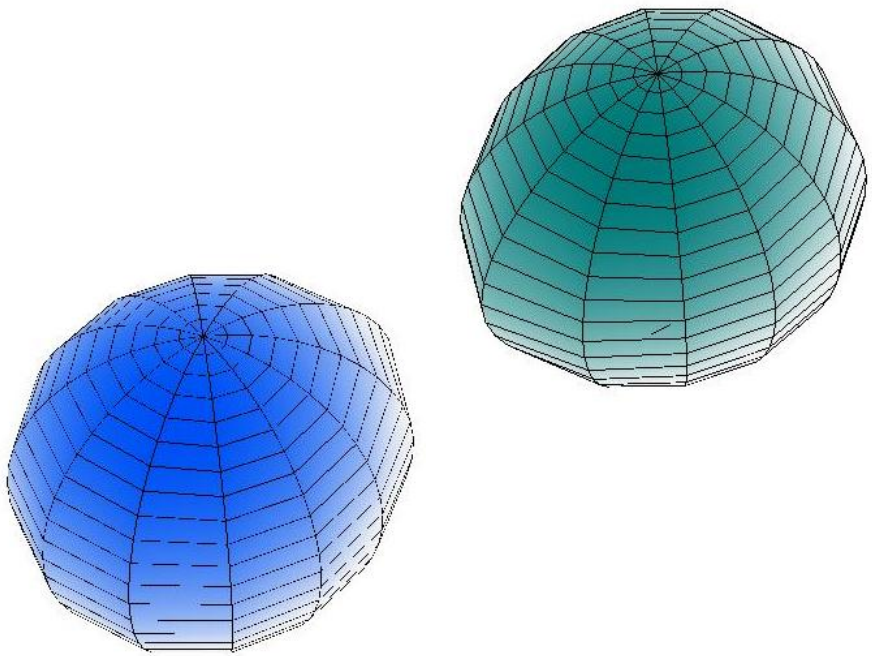


$\Rightarrow$ The quasi-stationary approach (Helical Killing Vector) allowed to compute location of the Innermost Stable Circular Orbit (ISCO) for two black holes.

At this orbit compact objects fall one on another and the gravitational wave signal is strongly reduced.

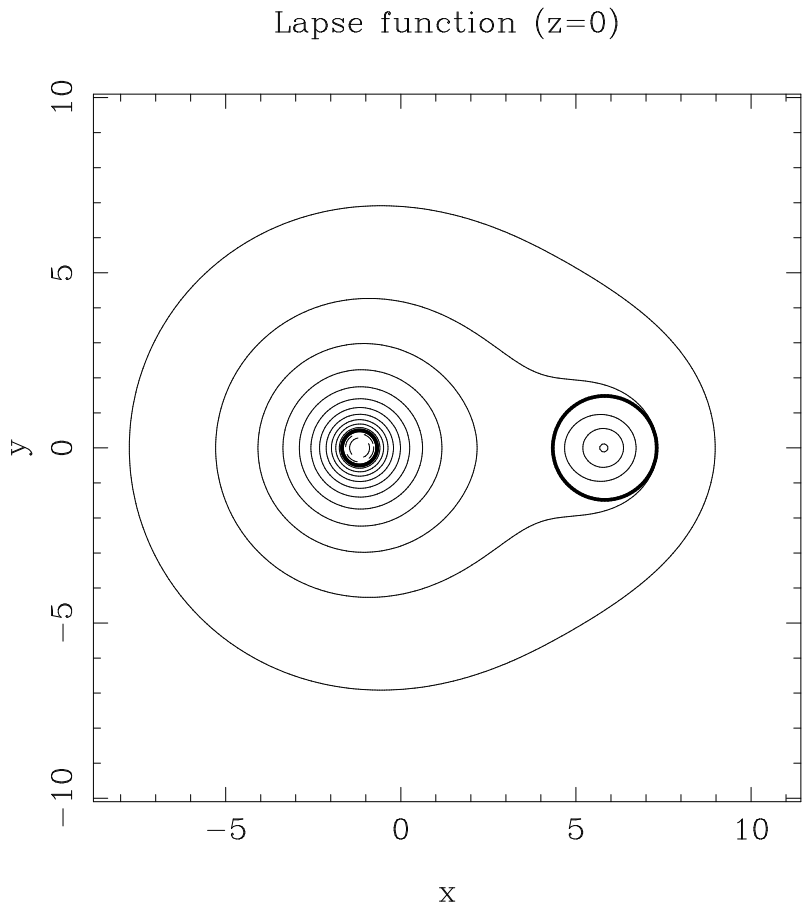
Irrotational configurations of binary neutron stars

Bonazzola *et al.* (1999)



- For neutron stars, irrotational fluid motion is considered
- Symmetry / $z = 0$ plane is assumed
- Each star is associated to a grid, adapting to its surface

Neutron star – black hole models



In a similar way, mixed binaries have been computed in quasi-stationary approach (Grandclément 2007).

⇒ Numerical models in CFC and helical Killing vector field used as initial data for binary coalescence simulations.

Evolution of a single neutron star:

oscillations

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Hydrodynamics in a conserved form

General relativistic hydrodynamics can be written as a flux-conservative first order hyperbolic system:

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\boldsymbol{U}}{\partial t}+\frac{\partial\sqrt{-g}\boldsymbol{F}^i}{\partial x^i}\right]=\boldsymbol{Q},$$

with

$$\boldsymbol{U}=\left(\begin{array}{ccc} D & = & \rho\Gamma=\rho/\sqrt{1-v_iv^i} \\ M_i & = & (e+p)\Gamma^2v_i \\ \tau & = & (e+p)\Gamma^2-p-D \end{array}\right)$$ the conserved variables,

expressed in terms of the primitive variables: (ρ, v_i, e, p).

$\Rightarrow$ *High-Resolution Shock-Capturing* schemes: good treatment of shocks (Valencia formulation)

$\Rightarrow$ but large CPU cost (recovery of primitive variables)

Use of primitive variables

$\Rightarrow$ Use of equations directly evolving primitive variables:

$$
\begin{aligned}
\partial_t U_i \quad &= -v^j D_j U_i + U_j D_i \beta^j - D_i N + U_i U^j D_j N \\
&+ \frac{c_s^2 N U_i}{\Gamma^2 - c_s^2(\Gamma^2-1)} (D_j U^j - K) + U_i \frac{\Gamma^2(c_s^2-1)}{\Gamma^2 - c_s^2(\Gamma^2-1)} N U^l U^j K_{lj} \\
&- \frac{N}{\Gamma^2} \left(D_i H - \frac{\Gamma^2(1-c_s^2)}{\Gamma^2 - c_s^2(\Gamma^2-1)} U_i U^j D_j H \right) \\
\partial_t H \quad &= -v^i D_i H - c_s^2 \frac{\Gamma^2 N}{\Gamma^2 - c_s^2(\Gamma^2-1)} \left(K_j U^i U^j - K + D_i U^i \right. \\
&\left. - \frac{U^i}{\Gamma^2} \left(D_i H - e^{-H} \theta D_i S \right) \right) \\
\partial_t S \quad &= -v^i D_i S - \frac{N}{\Gamma \theta n_B} \left(Q + \mu_l \sigma \right) .
\end{aligned}
$$

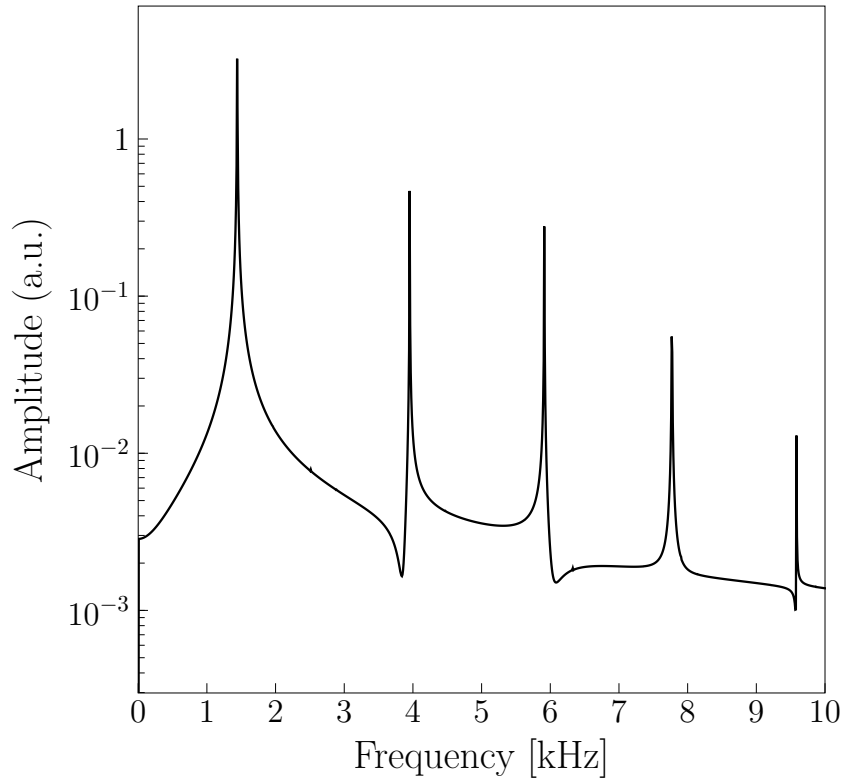
In conjunction with the use of spectral methods, large gain in execution time (Cf. Servignat *et al.* 2023).

$\Rightarrow$ no possibility of having shocks ...

Spherical symmetry:

frequencies

- Spherical TOV initial data
- Polytropes and APR EoSs
- Perturbed by radial velocity or density profile
- Cowling (no evolution of gravitational field) or full dynamics

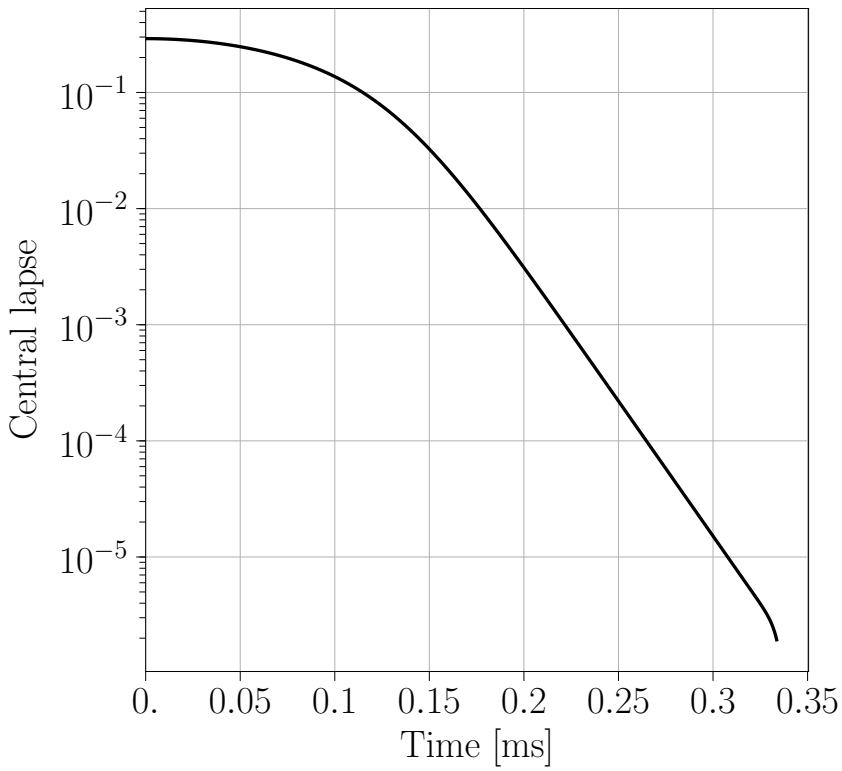


$\Rightarrow$ Recovery of frequencies from previous works in all cases (Servignat *et al.* 2023).

$\Rightarrow$ Tests with "smooth" representations of realistic 1- and 2-arguments EoS (n_B, Y_e) (Servignat *et al.* 2024).

Spherical symmetry:

collapse to a black hole



In case initial data on the unstable branch:

$\Rightarrow$ Collapse up to the formation of a black hole:

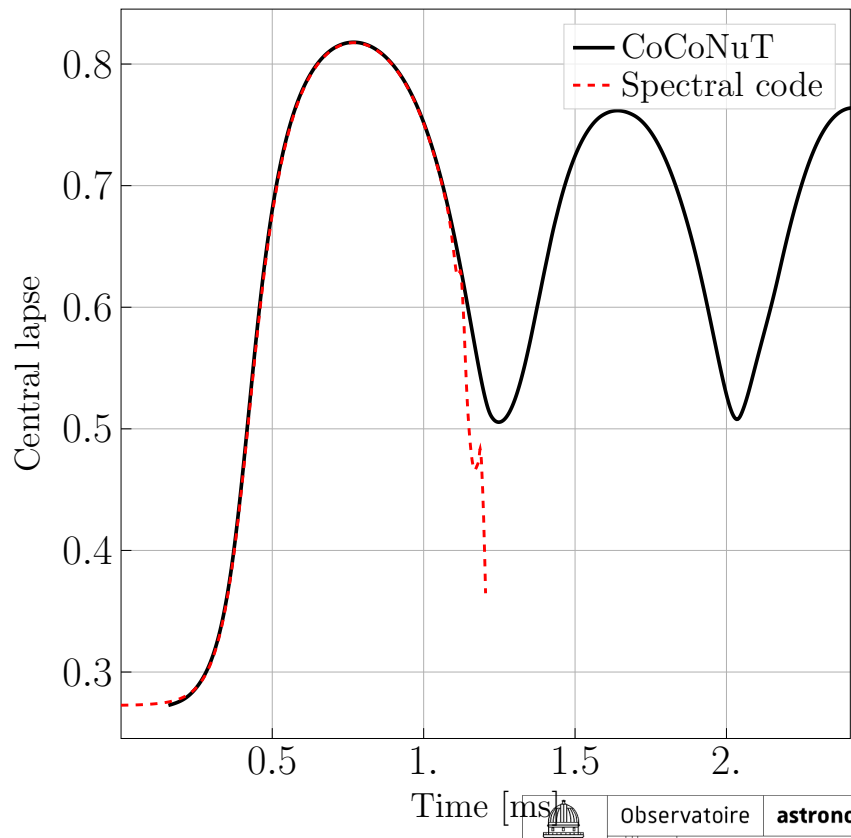
- Collapse of the lapse in the center (maximal slicing)
- Apparent horizon detected

$\Rightarrow$ Fully non-linear code: not only perturbations.

Limitations: migration test

In half of cases the star on the unstable branch is not collapsing to a black hole:
$\Rightarrow$ expected migration to the stable branch and oscillations around stable configuration

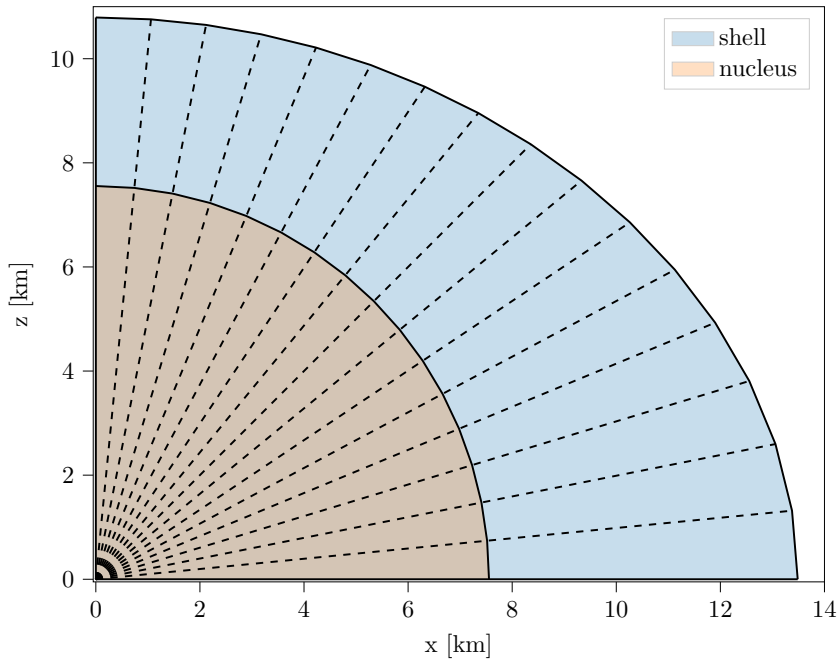
- The code does not converges any more after ~ 1.1 ms.
- Formation of a shock, which cannot be resolved with spectral methods ...
- ... nor with primitive variable formulation.
- Comparison with HRSC code shows perfect agreement until the shock appears.



3D code: adaptive grid

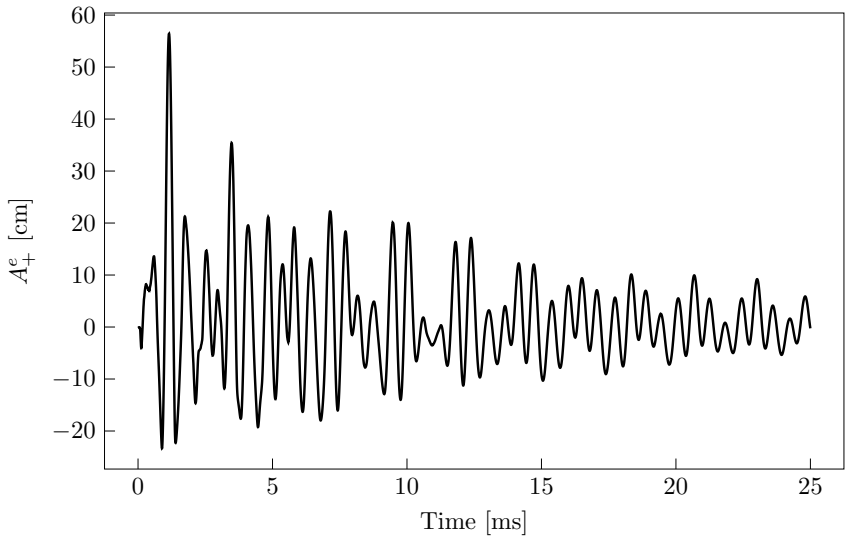
Beyond spherical and axial symmetries: assume only symmetry / equatorial plane and $(x, y) \mapsto (-x, -y)$.

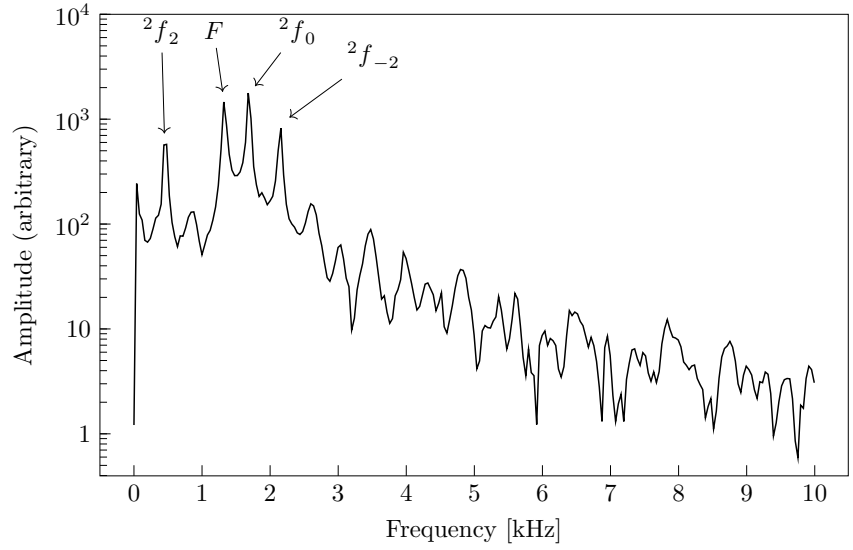
- Grid describing the star is adapted its surface
- Mapping numerical $\rightarrow$ physical coordinates
- Inner non-deformed domain (nucleus) to deal with coordinate singularity at $r = 0$
- Metric equations (CFC) solved in entire space
- Interpolation between both grids



$\Rightarrow$ Code Roxas (Servignat & Novak 2025)

3D code: gravitational waves





- Conformal-Flatness-Condition $\Rightarrow$ gravitational waves extracted through modified quadrupole formula
- Spectral methods $\Rightarrow$ natural decomposition onto spherical harmonics, and study mode-by-mode
- Run on single-processor laptop (~ 2 days, though)

3D code: oscillations of rigidly rotating stars

Relative differences (in %) between the frequencies computed with Roxas and those reported in the literature

- Polytropic equation of state
- Sequence with the same central enthalpy
- Increasing rotation: 0 Hz (BU0) to 831 Hz (BU7)
- And gravitational mass 1.4 $M_\odot$ (BU0) to 1.665 $M_\odot$ (BU7)

Model	F		H_1		${}^2f_0$		${}^2f_2$		${}^2f_{-2}$	
	Cowling	CFC	Cowling	CFC	Cowling	CFC	Cowling	Full GR	Cowling	Full GR
BU0	0.15	0.83	0.00	0.38	0.48	0.63	0.00	0.83	0.00	0.83
BU1	0.11	0.70	0.20	0.82	0.27	0.12	0.51	0.09	0.09	0.56
BU2	0.12	1.15	0.54	0.59	0.90	0.43	0.98	0.48	0.09	0.10
BU3	0.35	1.25	0.14	0.15	0.63	0.65	0.09	0.63	0.61	0.24
BU4	0.67	1.36	0.20	0.40	0.64	0.99	1.66	1.11	0.57	0.09
BU5	0.64	0.08	0.07	0.15	0.38	0.52	2.69	4.64	0.44	1.44
BU6	0.93	3.10	0.20	0.30	1.82	0.86	2.01	12.3	0.40	1.82
BU7	0.70	0.42	0.02	0.57	1.31	1.77	11.0	×	1.88	3.94

Summary

- Stationary models of rotating neutron stars, with magnetic field: need for General Relativity and numerical tools.
- Quasi-stationary models of binary compact objects computed as initial data for evolution codes.
- Fast and flexible evolution code for isolated neutron stars: Roxas.

Not discussed here:

- More detailed codes for stationary rotating models: crust, stable magnetic field (poloidal-toroidal), superfluidity (glitches), ...
- Evolution codes for binary neutron stars: Einstein Toolkit (einsteintoolkit.org), Arepo (Lioutas *et al.* 2024), ...
- Alternative theories of gravity.

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